Additional Chapter 13 Problems

MA2160, Spring '07

For the following problem, let $u = \overrightarrow{PQ}$ and $v = \overrightarrow{PR}$, and find

(a). **the component forms of u and v**

(b). **the magnitude of v**

(c). $2u + v$

1. $P = (1, 2), Q = (4, 1), R = (5, 4)$

solution:
$P = (1, 2), Q = (4, 1), R = (5, 4)$
(a). $u = \overrightarrow{PQ} = (4 - 1)\hat{\imath} + (1 - 2)\hat{\jmath} = 3\hat{\imath} - \hat{\jmath}$, $v = \overrightarrow{PR} = (5 - 1)\hat{\imath} + (4 - 2)\hat{\jmath} = 4\hat{\imath} + 2\hat{\jmath}$
(b). $\|v\| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$
(c). $2u + v = (6\hat{\imath} - 2\hat{\jmath}) + (4\hat{\imath} + 2\hat{\jmath}) = 10\hat{\imath}$

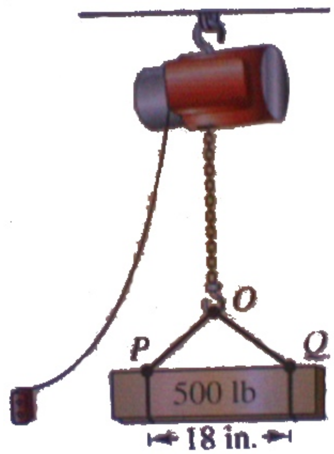
For the following problem, find the component form of the vector v given its magnitude and the angle it makes with the x-axis.

2. $\|v\| = \frac{1}{2}, \theta = 225°$

solution:

$$\begin{aligned} v = \|v\| \cos\theta\hat{\imath} + \|v\| \sin\theta\hat{\jmath} &= \frac{1}{2}\cos 225°\hat{\imath} + \frac{1}{2}\sin 225°\hat{\jmath} \\ &= -\frac{\sqrt{2}}{4}\hat{\imath} + \frac{\sqrt{2}}{4}\hat{\jmath} \end{aligned}$$

3. ***Minimum Length*** In a manufacturing process, an electric hose lifts 500-pound ingots. The length of the cable (see figure below). The length of the cable connecting points P, O, and Q is L inches. (Assume that O is at the midpoint of the cable.)



(a). Write the tension T in the cable as a function of L. What is the domain of the function?

(b). Use the function in part (a) to complete the table.

L	19	20	21	22	23	24	25
T							

(c). Use a graphing utility to graph the tension function.

(d). Find the shortest cable connecting points P, O, and Q that can be used if the tension in the cable cannot exceed 400 pounds.

(e). Find (if possible) $\lim_{L\to\infty} T$. Interpret the result in the context of the problem.

solution:

(a). The length of cable POQ is L.

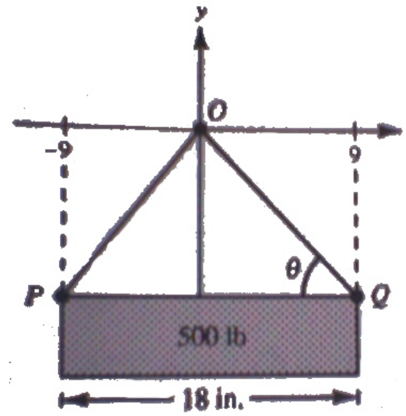
$$\overrightarrow{OQ} = 9\hat{\imath} + y\hat{\jmath}$$

$$L = 2\sqrt{9^2 + y^2} \Rightarrow \sqrt{\frac{L^2}{4} - 81} = y$$

Tension: $T = c\|\overrightarrow{OQ}\| = c\sqrt{81 + y^2}$

And, $cy = 250 \Rightarrow T = \frac{250}{y}\sqrt{81 + y^2} \Rightarrow T = \frac{250}{\sqrt{\frac{L^2}{4} - 81}} \cdot \frac{L}{2} = \frac{250L}{\sqrt{L^2 - 324}}$

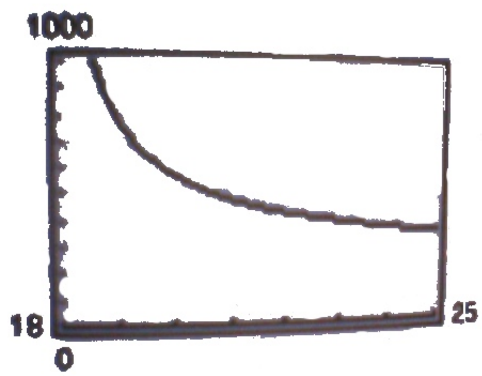
Domain: $L > 18$ inches.



(b)

L	19	20	21	22	23	24	25
T	780.9	573.54	485.36	434.81	401.60	377.96	360.24

(c).



(d). The line $T = 400$ intersects the curve at $L = 23.06$ inches.
(e). $\lim_{L\to\infty} T = 250$. The maximum tension is 250 pounds in each side of the cable since the total weight is 500 pounds.

4. Find the coordinates of the point located on the y-axis and 7 units to the left of the xz-plane.

solution:
$x = z = 0, y = -7; (0, -7, 0)$

For the following problem, determine the location of the point (x, y, z) such that the given condition is satisfied

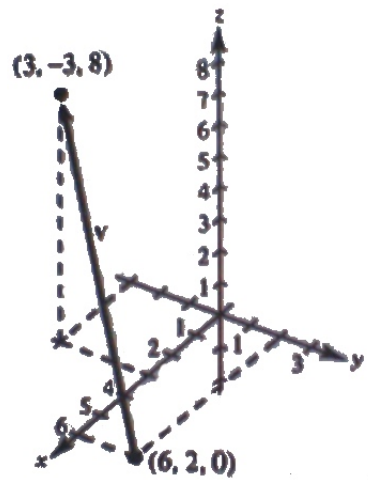
5. $xy < 0$

solution:
Looking towards the xy-plane from the positive z-axis. The point is either in the second quadrant $(x < 0, y > 0)$ or in the fourth quadrant $(x > 0, y < 0)$. The z-coordinate can be any number.

The initial point and terminal point of a vector are given. Sketch the directed line segment and find its component form of the vector.

6. Initial Point: $(6, 2, 0)$ and Terminal Point: $3, -3, 8)$

solution:
$v = \langle 3, -6, -3 - 2, 8 - 0 \rangle = \langle -3, -5, 8 \rangle$



For the following problem, use vectors to determine whether the points lie on a straight line.

7. $(5, -4, 7), (8, -5, 5), (11, 6, 3)$

solution:

$$\begin{aligned} v &= (8-5)\hat{\imath} + (-5+4)\hat{\jmath} + (5-7)\hat{k} = 3\hat{\imath} - 1\hat{\jmath} - 2\hat{k} \\ w &= (11-5)\hat{\imath} + (6+4)\hat{\jmath} + (3-7)\hat{k} = 6\hat{\imath} + 10\hat{\jmath} - 4\hat{k} \end{aligned}$$

Since v and w are not parallel, the points do not lie on the same line.

8. Find a vector v of magnitude 8 in the direction $6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$.

solution:
$8\frac{6\hat{\imath}-3\hat{\jmath}+2\hat{k}}{\sqrt{49}} = -\frac{8}{7}(6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}) = \frac{48}{7}\hat{\imath} - \frac{24}{7}\hat{\jmath} + \frac{16}{7}\hat{k}$

Let $u = \overrightarrow{PQ}$ and $v = \overrightarrow{PR}$ and find

(a). the component forms of u and v

(b). $u \cdot v$

(c). $v \cdot v$

9. $P = (2, -1, 3), Q = (0, 5, 1), R = (5, 5, 0)$

solution:
(a).

$$\begin{aligned} u &= \overrightarrow{PQ} = -2\hat{\imath} + 6\hat{\jmath} - 2\hat{k} \\ v &= \overrightarrow{PR} = 3\hat{\imath} + 6\hat{\jmath} - 3\hat{k} \end{aligned}$$

(b). $u \cdot v = (-2)(3) + (6)(6) + (-2)(-3) = 36$
(c). $v \cdot v = 9 + 36 + 9 = 54$

Determine whether the vectors are orthogonal, parallel, or neither.

10. $-4\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$, $16\hat{\imath} - 12\hat{\jmath} + 24\hat{k}$

solution:
$u = -4\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$, $v = 16\hat{\imath} - 12\hat{\jmath} + 24\hat{k}$
Since $v = -4u$, the vectors are parallel.

For the following, find the angle θ between the vectors u and v.

11. $u = 4\hat{\imath} - \hat{\jmath} + 5\hat{k}$, $v = 3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

    solution:
    $u \cdot v = 0 \Rightarrow u$ is orthogonal to v.
    $\theta = \frac{\pi}{2}$

12. $u = \hat{\imath} - 3\hat{k}$, $v = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$

    solution:
    $u \cdot v = -1$

$$\begin{aligned} \|u\| &= \sqrt{10} \\ \|v\| &= 3 \end{aligned}$$

    $\cos\theta = \frac{|u \cdot v|}{\|u\|\|v\|} = \frac{1}{3\sqrt{10}}$
    $\Rightarrow \theta \approx 83.9°$

13. ***Work*** An object is pulled 8 feet across a floor using a force of 75 pounds. Find the work done if the direction of the force is 30° above the horizontal.

solution:

$$\begin{aligned} W &= \mathbf{F} \cdot \overrightarrow{PQ} \\ &= \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos\theta \\ &= (75)(8)\cos 30^\circ \\ &= 300\sqrt{3} \text{ ft} \cdot \text{lb} \end{aligned}$$

For the following, let $u = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$, $v = 2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$, and $w = -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

14. Find the angle between u and v.

    solution:

$$\begin{aligned} \cos\theta &= \frac{|u \cdot v|}{\|u\|\|v\|} = \frac{11}{\sqrt{14}\sqrt{29}} \\ \theta &= \arccos\left(\frac{11}{\sqrt{14}\sqrt{29}}\right) \approx 56.9^\circ \end{aligned}$$

15. Find the work done in moving an object along the vector u if the applied force is w.

    solution:
    Work$=|u \cdot w| = |-3-4+2| = 5$

For the following problem

(a). find the projection of u onto v

(b). find the vector component orthogonal to v

16. $u = \hat{\imath} + 4\hat{k}$, $v = 3\hat{\imath} + 2\hat{k}$

solution:
(a). $w_1 = \left(\frac{u \cdot v}{\|v\|^2}\right) v = \frac{11}{3}(3\hat{\imath} + 2\hat{k}) = \frac{33}{13}\hat{\imath} + \frac{22}{13}\hat{k}$
(b). $w_2 = u = w_1 = (\hat{\imath} + 4\hat{k}) - (\frac{33}{13}\hat{\imath} + \frac{22}{13}\hat{k}) = -\frac{20}{13}\hat{\imath} + \frac{30}{13}\hat{k}$

For the following problems let $u = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$**,** $v = 2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$**, and** $w = -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$**.**

17. Show that $u \times v = -(v \times u)$.

    solution:

    $$u \times v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\hat{\imath} + 11\hat{\jmath} - 8\hat{k}$$

    $$v \times u = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & -3 \\ 3 & -2 & 1 \end{vmatrix} = -10\hat{\imath} - 11\hat{\jmath} + 8\hat{k}$$

    $\Rightarrow u \times v = -(v \times u)$.

18. Show that $u \times (v + w) = (u \times v) + (u \times w)$.

    solution:

    $$u \times (v + w) = (3\hat{\imath} - 2\hat{\jmath} + \hat{k}) \times (\hat{\imath} - 2\hat{\jmath} - \hat{k}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath} - 4\hat{k}$$

    $$u \times v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\hat{\imath} + 11\hat{\jmath} - 8\hat{k}$$

    $$u \times w = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = -6\hat{\jmath} - 7\hat{\jmath} + 4\hat{k}$$

    $(u \times v) + (u \times w) = 4\hat{\imath} + 4\hat{\jmath} - 4\hat{k} = u \times (v + w)$

19. Find the area of the triangle with adjacent sides v and w.

    solution: Area of triangle $= \frac{1}{2}\|v \times w\| = \frac{1}{2}\sqrt{(-2)^2 + (-1)^2} = \frac{\sqrt{5}}{2}$

20. ***Volume*** Use the triple scalar product to find the volume of the parallelepiped with edges $u = 2\hat{\imath} + \hat{\jmath}$, $v = 2\hat{\jmath} + \hat{k}$ and $w = -\hat{\jmath} + 2\hat{k}$

    solution:

    $$V = |u \cdot (v \times w)| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2(5) = 10$$

**This is just a collection of practice problems. It does not represent what may or may not be on a test. Undoubtedly, there are topics not covered in the collection of problems.*